ANALYSIS OF INVERSE HEAT-CONDUCTION PROBLEMS BY MEANS OF SIMILARITY SOLUTIONS

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Similarity solutions are given for a linear heat-conduction equation for a wide class of boundary conditions. The solutions are used for analyzing inverse problems that are applied to the determination of external heating conditions.

In recent years many works have been devoted to studying inverse heat-conduction problems that are applied to the determination of external heating conditions. A fairly complete survey of publications in this area is contained in [1-3]. We attempt here to unify a class of inverse heat-conduction problems permitting similarity solutions.

As is known from [6], a linear heat-conduction equation with boundary conditions corresponding to a semibounded region with a boundary that is instantaneously heated to a constant temperature has a similarity solution:

$$a\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial t},\quad u=u_0\,\mathrm{erfc}\,\xi,\quad \xi=\frac{x}{2\sqrt{at}}. \tag{1}$$

It is shown in [7] that the known class of solutions [5] for the linear heat-conduction equation with boundary conditions

$$u(x,\ 0)=u(\infty,\ t)=0,\quad u(0,\ t)=\sum_0^n\alpha_k t^{\gamma_k},\ \alpha_k=\mathrm{const},\ \gamma_k=\mathrm{const} \tag{2}$$

is self-similar and can be represented for $\gamma_k = k/2$ as follows:

$$u=\sum_0^n\alpha_k t^{\frac{k}{2}}P_{2k}(\xi)\int_\infty^\xi\frac{\exp\frac{z^2}{4}}{P_{2k}^2(z)}dz=\sum_0^n\alpha_k t^{\frac{k}{2}}\frac{i^k\,\mathrm{erfc}\,\xi}{i^k\,\mathrm{erfc}\,0}. \tag{3}$$

We study the self-similar variable ξ from Eqs. (1) in more general form:

$$\xi=\frac{x-x_0}{2\sqrt{a(t-t_0)}},\quad x_0=\mathrm{const},\quad t_0=\mathrm{const}. \tag{4}$$

This form of the self-similar variable allows us to broaden [8] the class of boundary conditions permitting similarity solutions for the heat-conduction equation, after we have supplemented it with the conditions

$$u(0,\ t)=\sum_0^n\alpha_k t^{\frac{k}{2}}i^k\,\mathrm{erfc}\,\frac{x_0}{2\sqrt{at}} \tag{5}$$

and

$$u(0,\ t)=\begin{cases}u_1(0,\ t)=\sum_0^{n_1}\alpha_{1,k}t^{\frac{k}{2}}, & 0<t<t_1\\ u_2(0,\ t)=u_1(0,\ t)+\sum_0^{n_2}\alpha_{2,k}(t-t_1)^{\frac{k}{2}}, & t_1\leqslant t<t_2\\ u_m(0,\ t)=u_{m-1}(0,\ t)+\sum_0^{n_m}\alpha_{m,k}(t-t_{m-1})^{\frac{k}{2}}, & t_{m-1}\leqslant t\leqslant t_m\end{cases} \tag{6}$$

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In the case of (5) and (6) the corresponding solutions take the form

$$u = \sum_{0}^{n} \alpha_k t^{\frac{k}{2}} i^k \operatorname{erfc} \frac{x + x_0}{2\sqrt{at}}, \tag{7}$$

$$u = \sum_{j=1}^{m} \sum_{k=0}^{n_j} \alpha_{j,k} (t - t_{j-1})^{\frac{k}{2}} \frac{i^k \operatorname{erfc} \dfrac{x}{2\sqrt{a(t-t_{j-1})}}}{i^k \operatorname{erfc} 0},$$

$$t < t_j \to \alpha_{jk} = 0. \tag{8}$$

A characteristic feature of the similarity solutions under study is their universality, since they essentially comprise only one class of the well-studied functions i^kerfc [4]; in addition, a very simple relation of temperature and heat flow is obtained for these solutions on the boundary x = 0:

$$u(0,\ t) = \sum_{0}^{n} \alpha_k t^{\frac{k}{2}},\ -\lambda \frac{\partial u(0,\ t)}{\partial x} = \sum_{0}^{n} \beta_k t^{\frac{k-1}{2}},$$

$$\beta_k = \frac{-\lambda (i^k \operatorname{erfc})'_0}{2\sqrt{a}\, i^k \operatorname{erfc} 0} \alpha_k. \tag{9}$$

The circumstances indicated enable us to use effectively the similarity solutions for analyzing inverse heat-conduction problems that are applied to the determination of external heating conditions.

We assume that the temperature variation in time is known at a certain point x = b of a semi-infinite body x > 0, whose initial temperature is equal to 0, $u(b,\ t) = \varphi(t)$. We must determine the conditions of external heating $u(0,\ t)$; $-\lambda[\partial u(0,\ t)/\partial x]$. We assume that the variation of the surface temperature can be represented as follows:

1. The temperature for x = 0 varies according to Eq. (2); thus, due to (3)

$$\varphi(t) = \sum_{0}^{n} \alpha_k t^{\frac{k}{2}} \frac{i^k \operatorname{erfc} \dfrac{b}{2\sqrt{at}}}{i^k \operatorname{erfc} 0}. \tag{10}$$

Thus, the problem reduces to an approximation of the experimental curve $\varphi(t)$ in the form of (9), i.e., to the solution of a corresponding system of linear equations for determining the unknowns α_k, k = 0, 1, ..., n. This case is studied in [9] by a somewhat different procedure.

2. The temperature at point x = 0 varies according to Eq. (5); thus, due to (7)

$$\varphi(t) = \sum_{0}^{n} \alpha_k t^{\frac{k}{2}} i^k \operatorname{erfc} \frac{x_0 + b}{2\sqrt{at}} \tag{11}$$

and α_k is determined in the same manner as in the preceding case.

3. The temperature at point x = 0 varies according to Eq. (6); thus, due to (8)

$$\varphi(t) = \sum_{j=0}^{m} \sum_{k=0}^{n_j} \alpha_k (t - t_{j-1})^{\frac{k}{2}} \frac{i^k \operatorname{erfc} \dfrac{b}{2\sqrt{a(t-t_{j-1})}}}{i^k \operatorname{erfc} 0}. \tag{12}$$

In this case the problem is solved successively. First of all, we determine $\alpha_{1,k}$ in a manner similar to the preceding case, substituting into (7) the $(1 + n_1)$ values of t from the interval $(0\text{-}t_1)$ and solving the corresponding system; then we determine $\alpha_{2,k}$ in the interval $(t_1\text{-}t_2)$, taking into account the already determined functions $\alpha_{1,k}$, etc. The particular case of condition (5), when $u_j(0,\ t) = \text{const}$, is studied in [1, 10].

We find the heat flow in all the cases under study according to Eqs. (9).

Below we estimate the error for the solution of the inverse problem, which is related to the representation of the solution in similarity form. We assume that the varying temperature of surface f(t) can be estimated according to the equation

$$\sum_{0}^{n} \bar{\alpha}_k t^{\frac{k}{2}} \leqslant f(t) \leqslant \sum_{0}^{n} \bar{\bar{\alpha}}_k t^{\frac{k}{2}}.$$

Thus, on the basis of the similarity representation for solution (3), we can write

$$\bar{u}(b,\ t) = \sum_{0}^{n} \bar{\alpha}_k t^{\frac{k}{2}} \frac{i^k \operatorname{erfc} \frac{b}{2\sqrt{at}}}{i^k \operatorname{erfc} 0}, \tag{13}$$

$$\bar{\bar{u}}(b,\ t) = \sum_{0}^{n} \bar{\bar{\alpha}}_k t^{\frac{k}{2}} \frac{i^k \operatorname{erfc} \frac{b}{2\sqrt{at}}}{i^k \operatorname{erfc} 0} \tag{14}$$

and, evidently,

$$\bar{u}(b,\ t) \leqslant u(b,\ t) \leqslant \bar{\bar{u}}(b,\ t). \tag{15}$$

By subtracting (13) from (14), we obtain

$$\bar{\bar{u}}(b,\ t) - \bar{u}(b,\ t) = \sum_{0}^{n} (\bar{\bar{\alpha}}_k - \bar{\alpha}_k)\, t^{\frac{k}{2}} \frac{i^k \operatorname{erfc} \frac{b}{2\sqrt{at}}}{i^k \operatorname{erfc} 0}. \tag{16}$$

If we assume that $\bar{\bar{\alpha}}_k = \bar{\alpha}_k$ for $k \neq 0$ and for $k = 0$ $\bar{\bar{\alpha}}_0 = \bar{\alpha}_0 + \varepsilon_1$, then

$$\bar{\bar{u}}(b,\ t) - \bar{u}(b,\ t) = \varepsilon_1\, i^0 \operatorname{erfc} \frac{b}{2\sqrt{at}}$$

and

$$\varepsilon_1 \leqslant \frac{\varepsilon(b)}{\operatorname{erfc} \frac{b}{2\sqrt{at}}},$$

where

$$\varepsilon(b) = \sup\,[\bar{\bar{u}}(b,\ t) - \bar{u}(b,\ t)].$$

NOTATION

a, thermal conductivity; t, time; m, n, maximum number of summation terms; u, temperature; x, coordinate; z, integration variable; α, β, coefficients; γ, exponent; ε, error; λ, thermal-conductivity coefficient; ξ, similarity variable; i, j, k, summation indices.

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